Section 1.5 Solution Sets of Linear Systems

Homogeneous Linear Systems

A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Remark: Such a system $A\mathbf{x} = \mathbf{0}$ always has at least one solution, namely $\mathbf{x} = \mathbf{0}$ (the zero vector in $\mathbb{R}^n$). This zero solution is usually called the **trivial solution**.

Given $A\vec{x}=\vec{0}$, an important question is if there exist a nontrivial solution, i.e, a nonzero verctor $\vec{x}$ such that $A\vec{x}=\vec{0}$.

By the Existence and Uniqueness Thm in Sec. 1.2, we have.

Theorem

The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.

Example 1. Determine if the following homogeneous system has a nontrivial solution. Try to use as few row operations as possible.

$$\begin{aligned} 2x_1 - 5x_2 + 8x_3 &= 0 \\ -2x_1 - 7x_2 + x_3 &= 0 \\ 4x_1 + 2x_2 + 7x_3 &= 0 \end{aligned}$$

ANS: The augmented matrix

$$\left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ -2 & -7 & 1 & 0 \\ 4 & 2 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 12 & -9 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & -12 & 9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The varible x_3 is free. Thus the system has a nontrival solution.

To solve the system (if the question asks).

$$\sim \left[\begin{array}{ccc|c} 2 & -5 & 8 & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 0 & \frac{17}{4} & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{17}{8} & 0 \\ 0 & 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\begin{cases} x_1 + \frac{17}{8} x_3 = 0 \\ x_2 - \frac{3}{4} x_3 = 0 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{17}{8} x_3 \\ x_2 = \frac{3}{4} x_3 \\ x_3 \text{ is free.} \end{cases}$

As a vector, the general solution to $A\vec{x} = \vec{0}$ has the form

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{17}{8} x_3 \\ \frac{3}{4} x_3 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -\frac{17}{8} \\ \frac{3}{4} \\ 1 \end{bmatrix}}_{\vec{v}} = x_3 \vec{v}$$

Geometrically, the solution set is a line through $\vec{0}$ and $\vec{v}$ in $\mathbb{R}^3$.

Example 2. Describe all solutions of the homogeneous system:

$$2x_1 - x_2 - 3x_3 = 0 \Rightarrow x_1 = \frac{1}{2}x_2 + \frac{3}{2}x_3$$

ANS:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 + \frac{3}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{3}{2}x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} + x_3 \underbrace{\begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}}_{\vec{v}}$$

(x_2 and x_3 are any real numbers).

Note $\vec{u}$, $\vec{v}$ are not a scalar multiple of each other.

So the solution is a plane in $\mathbb{R}^3$ spanned by

$$\vec{u} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

Parametric Vector Form

Whenever a solution set is described explicitly with vectors as in **Example 1** or **2**, we say that the solution is in **parametric vector form**.

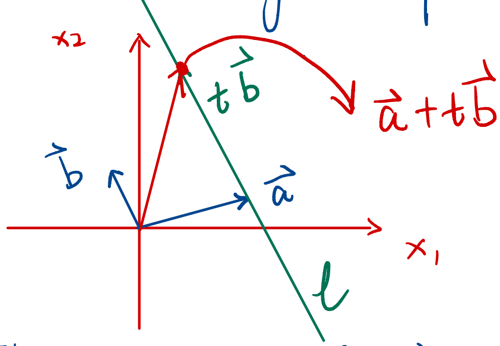
Sometimes, the free parameters are denoted by s, t etc. as to emphasize that the parameters vary over all real numbers.

Example 3. Find a parametric equation of the line M through $\mathbf{p}$ and $\mathbf{q}$.

$$\mathbf{p} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

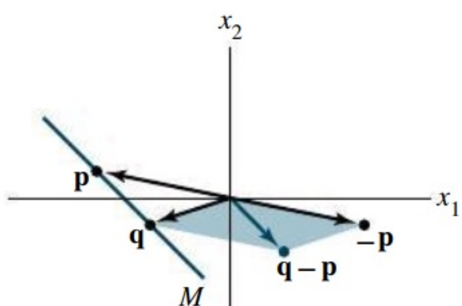
ANS: We need to know the follow two general facts:

① The line through $\vec{a}$ parallel to $\vec{b}$ can be written as $\vec{a} + t\vec{b}$:



i.e. any point on the line ℓ is $\vec{a} + t\vec{b}$ for some value of t.

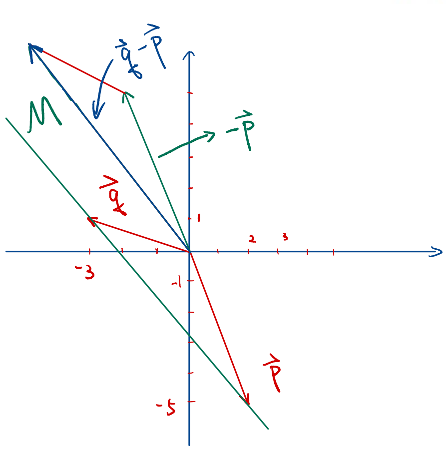
② The line M through $\vec{p}$ and $\vec{q}$ is parallel to the vector $\vec{q} - \vec{p}$:



The line through **p** and **q**.

Thus M is the line through $\vec{p}$ parallel to $\vec{q} - \vec{p}$.

So the parametric equation for M is



$$\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$$

$$\Rightarrow \vec{x} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} + t \begin{bmatrix} -5 \\ 6 \end{bmatrix}$$

Solutions of Nonhomogeneous Systems

When a nonhomogeneous linear system has many solutions, the general solution can be written in parametric vector form as one vector plus an arbitrary linear combination of vectors that satisfy the corresponding homogeneous system.

Example 4. Describe the solutions of the following system in parametric vector form. Also, give a geometric description of the solution set.

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 4 \\ x_1 + 4x_2 - 8x_3 &= 7 \\ -3x_1 - 7x_2 + 9x_3 &= -6 \end{aligned}$$

In Handwritten HW#6, you need to show the general solution to the corresponding homogenous system

$$\begin{aligned} x_1 + 3x_2 - 5x_3 &= 0 \\ x_1 + 4x_2 - 8x_3 &= 0 \\ -3x_1 - 7x_2 + 9x_3 &= 0 \end{aligned}$$ is $\mathbf{x} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ (in the parametric form).

Row reduce the augmented matrix for the system:

$$\left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow \begin{cases} x_1 + 4x_3 = -5 \\ x_2 - 3x_3 = 3 \\ 0 = 0 \end{cases} \Rightarrow \begin{cases} x_1 = -5 - 4x_3 \\ x_2 = 3 + 3x_3 \\ x_3 \text{ is free.} \end{cases}$$

In parametric form

$$\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}-5-4x_3\\3+3x_3\\x_3\end{bmatrix} = \underbrace{\overset{\vec{a}}{\begin{bmatrix}-5\\3\\0\end{bmatrix}}}_{=\vec{p}} \overset{+}{+} x_3 \underbrace{\overset{\vec{b}}{\begin{bmatrix}-4\\3\\1\end{bmatrix}}}_{=\vec{v}_h}$$

The solution set is the line through $\begin{bmatrix}-5\\3\\0\end{bmatrix}$. parallel to the line that is the solution set of the corresponding to the homogeneous system $A\vec{x}=\vec{0}$. This result is true in general.

Theorem 6

Suppose the equation $A\mathbf{x} = \mathbf{b}$ is consistent for some given $\mathbf{b}$, and let $\mathbf{p}$ be a solution. Then the solution set of $A\mathbf{x} = \mathbf{b}$ is the set of all vectors of the form $\mathbf{w} = \mathbf{p} + \mathbf{v}_h$, where $\mathbf{v}_h$ is any solution of the homogeneous equation $A\mathbf{x} = \mathbf{0}$.

Summary: Writing a solution set (of a consistent system) in parametric vector form

1. Row reduce the augmented matrix to reduced echelon form.
2. Express each basic variable in terms of any free variables appearing in an equation.
3. Write a typical solution $\mathbf{x}$ as a vector whose entries depend on the free variables, if any.
4. Decompose $\mathbf{x}$ into a linear combination of vectors (with numeric entries) using the free variables as parameters.

The following two questions are left as exercises. I will provide the complete notes for solving them after the lecture.

Exercise 5. Construct a 3×3 nonzero matrix A such that the vector $\begin{bmatrix}1\\1\\1\end{bmatrix}$ is a solution of $A\mathbf{x} = \mathbf{0}$

ANS: Look for $A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$ such that $1 \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_2 + 1 \cdot \mathbf{a}_3 = \mathbf{0}$. That is, construct A so that the sum of the columns is a zero vector. For example:

$$A = \begin{bmatrix}1 & 1 & -2\\1 & 2 & -3\\1 & -1 & 0\end{bmatrix}.$$

Exercise 6. Describe all solutions of $A\mathbf{x} = \mathbf{0}$ in parametric vector form, where A is row equivalent to the given matrix.

$$\begin{bmatrix} 1 & -2 & -9 & 5 \\ 0 & 1 & 2 & -6 \end{bmatrix}$$

ANS: $\begin{bmatrix} 1 & -2 & -9 & 5 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & -7 & 0 \\ 0 & 1 & 2 & -6 & 0 \end{bmatrix}.$ $\begin{aligned} x_1 - 5x_3 - 7x_4 &= 0 \\ x_2 + 2x_3 - 6x_4 &= 0 \end{aligned}$

The basic variables are x_1 and x_2, with x_3 and x_4 free. Next, $x_1 = 5x_3 + 7x_4$ and $x_2 = -2x_3 + 6x_4$. The general solution in parametric vector form is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 + 7x_4 \\ -2x_3 + 6x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 7x_4 \\ 6x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 6 \\ 0 \\ 1 \end{bmatrix}.$$